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QUANTUM ELECTRODYNAMICS IN MATERIAL MEDIA

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Synopsis

The canonical quantum theory of an electromagnetic field within an isotropic, nondispersive dielectric body in motion is developed. The quantization of the free field is carried out in two different ways, both by basing the canonical procedure directly on the field Lagrangian in the medium in a conventional way, and by using a new transformation procedure which maps the results from the vacuum field into those of the medium field. In the latter case, we also permit the existence of a family of covariant gauges. Covariant polarization vectors are introduced which are convenient for the relativistic appearance of the theory; in particular, the Gupta-Bleuler procedure will thereby involve only Lorentz invariant operator components and state vectors.

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1. Introduction

The classical electromagnetic radiation field within a transparent medium may conveniently be described by a phenomenological theory, so that the influence of the medium is taken into account by means of a refractive index n which in general is a function of both position and frequency. It should be expected that this approximative kind of description is good if the wavelength of the radiation is considerably greater than the interatomic distances in the material. This is however the case for a solid or liquid medium in the visible part of the spectrum. Further, in considering field quantities like energy and momentum one should employ Minkowski's energy-momentum tensor for the field, since this divergence-free tensor provides the simplest and most efficient description of the various optical phenomena. In this connection we may refer to the Jones-Richards experiment[1] involving a measurement of the electromagnetic radiation pressure exerted on a metal plate immersed in a dielectric liquid at rest. Minkowski's tensor leads also to a straightforward explanation of various experiments involving the propagation of light through media in motion, such as the Fizeau experiment[2] for rectilinear motion and the Sagnac-type experiment due to Heer, Little and Bupp[3] for rotational motion. (An extensive classical treatment of various aspects of the alternative electromagnetic energy-momentum tensors has recently been given by one of the authors[4].) The theory can readily be extended to the case where extraneous charges or currents are present. A typical example of such a situation is the Čerenkov effect; also in this case it has turned out that the agreement between the phenomenological theory and the experiments is remarkable[5].

The intention of the present paper is to give, still within the phenomenological kind of approach, a quantal description of the electromagnetic field within an infinite medium in interaction with charged particles in the general case where the medium moves with uniform velocity. Since the field equations can readily be obtained from a Lagrangian, it is natural to make use of the canonical formalism to construct the quantum theory. It can be verified that the results one obtains with the canonical procedure are closely connected with the results obtained with Minkowski's tensor. In this quantum theory the refractive index will of course appear as a parameter. Of earlier works on the relativistic phenomenological quantum electrodynamics the best known seems to be the extensive and excellent, but now somewhat old, treatment by J. M. Jauch and K. M. Watson(6). Related treatments have been given by V. L. Ginzburg(7), R. T. Cox(8), K. Nagy(9), M. I. Riazanov(10), C. Muzikař(11), R. Dobbertin(12), V. N. Tsytovich(13) and others. We shall occasionally refer to some of these works later on. As regards the classical phenomenological theory of radiating particles within a dielectric medium, this theory has been extensively studied by several authors concerned mainly with the relativistic theory(14), while other authors have concentrated on the microscopic non-relativistic quantal description(15).

Let us next enunciate the physical assumptions inherent in the following calculations. The medium is taken to be isotropic and infinite, so that the spatial dispersion is zero, but we shall also neglect the spectral dispersion and simply put the refractive index n equal to a real constant. While the influence from the medium on the photons is thus described by means of the refractive index, we shall on the other hand neglect any influence from the medium on the wave function for an electron and simply describe the latter by the usual Dirac equation. This is a reasonable assumption since the Compton wave length for the electron is $h/(mc) = 0.024$ Å, which is much smaller than the interatomic distances (see also the discussion by Cox(8)).

Now we know that a charged particle, when passing through matter, ionizes the atoms (or excites them to discrete energy levels) and thereby loses energy. As a consequence of the retardation, the particle emits bremsstrahlung. By working with heavy particles, the latter effect can be made very small. We shall however exclude the ionisation loss from the consideration, in accordance with our previous assumption about complete transparency (real n) so that the medium is not allowed to absorb photons. The only interaction between particle and medium that we shall consider is the interaction connected with emission or absorption of real or virtual "phenomenological" photons. Thus the only kind of dragging force (or stopping power) with which we shall be confronted in the case of a particle travelling through homogeneous matter is the force arising from the Čerenkov radiation. The Čerenkov loss only amounts to a small fraction of the total energy loss. JELLEY[5] reports that in a typical situation the actual ratio will be of the order of $1^0/_{00}$ (see also Ch. XII of Landau and Lifshitz's book[16]).

It is known that in the usual cases the modification introduced by the quantum theory is very small, owing to the smallness of the photon energy in comparison with the particle energy.* There are some situations however, in which the quantum theory may be of greater numerical importance: If we calculate higher order corrections to the S-matrix for general physical processes, particle collisions say, which take place in the medium, we have to integrate over the momenta of the virtual photons and have thus to include the long wavelength region for which the medium properties are important[10]. Apart from this, the phenomenological quantum electrodynamics represents from a formal point of view an interesting generalization of the conventional vacuum quantum electrodynamics.

We start in section 2 by developing the relativistic phenomenological theory in the configuration space of a free radiation field. The field is quantized according to the canonical scheme, and we base the calculation upon a Lagrangian density which represents the appropriate generalization of the Lagrangian density of conventional Fermi gauge vacuum quantum electrodynamics. The Lorentz covariance of the theory is examined by comparing the operators for momentum and angular momentum with the generating operators for translations and rotations in four-space.

In section 3 we go into Fourier space considerations. The four-potential is expanded in a form which is convenient for the relativistic appearance of the theory. The gauge condition is handled in a way which represents a generalization of the Gupta-Bleuler method for the vacuum field case. We find also in our case that there remain only two physically independent polarization directions of a photon of momentum $\boldsymbol{k}$; these polarization directions are however only orthogonal to the vector $\boldsymbol{k}$ in some special cases, viz. when $\boldsymbol{k}$ is parallel or antiparallel to the medium velocity $\boldsymbol{v}$ (or $\boldsymbol{v} = 0$). We close the section with some remarks upon the literature, especially in connection with the works of JAUCH and WATSON[6].

Section 4 is devoted to classical considerations. We introduce a new method by which the phenomenological theory of the medium field can be obtained from the vacuum field theory simply by a mapping procedure. Thereafter the gauge condition is examined. We find the interesting result that the polarization vectors $e_\mu^{(\lambda)}$, introduced earlier in section 3 for the

* If a fast particle travels through water, the modification to $\cos\theta$ (θ being the Čerenkov angle in the rest system) introduced by the quantum theory will amount to approximately 10^{-6}.

medium field, stand in intimate connection with the simplest gauge condition one can impose on the transformed vacuum field. Further, we make use of the fact that the rest frame of the medium represents a distinguished system of reference, to introduce a set of new *covariant* vectors $e_\mu^{(\lambda)}$.

In section 5 we make use of the mapping method from section 4 and construct the quantum theory of the medium field simply by transforming the results from the vacuum field theory. Besides, in the construction of the commutation rules in configuration space, we also permit a wider class of covariant gauges than the single Fermi gauge which is ordinarily used. With the use of the new covariant polarization vectors $e_\mu^{(\lambda)}$ the quantum theory is found to take a completely covariant form, both with respect to Lorentz transformations and with respect to the transformation medium-vacuum. In particular, the Gupta-Beuler method in this formulation only involves Lorentz invariant operator components and state vectors. Finally we write down the Feynman rules for the medium field.

The final section 6 contains some supplementing remarks to the foregoing. The possibility of dividing the total field angular momentum into an orbital part and a spin part is discussed. It is stressed that, although the canonical formalism is intimately connected with Minkowski's energy-momentum tensor, it does not thereby exclude the legitimacy of other expressions for the electromagnetic energy-momentum tensor. A proposal of a new experiment is put forward, consisting in a detection of spin in an electromagnetic wave travelling through a dielectric liquid. Such a detection would represent a further critical test of Minkowski's theory, in addition to the tests mentioned in the beginning of this section.

2. The Pure Radiation Field

It is convenient to start with a consideration of the pure radiation field. If we use units for which $c = 1$, Maxwell's equations can be written in any reference frame as

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0 \tag{2.1 a}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}, \quad \nabla \cdot \boldsymbol{D} = 0. \tag{2.1 b}$$

In this paper we shall employ the real coordinates $x_\mu = (x_0, x_1, x_2, x_3) = (t, \boldsymbol{x})$, the diagonal components of the metric tensor $g_{\mu\nu}$ in the form $(1, -1,$

$-1, -1)$, and shall put also $\hbar = 1$. If the antisymmetric field tensors $F_{\mu\nu}$ and $H_{\mu\nu}$ are defined by

$$(F_{10}, F_{20}, F_{30}) = \boldsymbol{E}, \quad (F_{23}, F_{31}, F_{12}) = -\boldsymbol{B}$$
$$(H_{10}, H_{20}, H_{30}) = \boldsymbol{D}, \quad (H_{23}, H_{31}, H_{12}) = -\boldsymbol{H},$$

we can write the field equations as

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{2.2 a}$$

$$\partial^\nu H_{\mu\nu} = 0, \tag{2.2 b}$$

where $\partial^\nu \equiv \partial/\partial x_\nu$. If the uniform four-velocity of the medium is denoted by $V_\mu = (V_0, \boldsymbol{V}) = \gamma(1, \boldsymbol{v})$, where $\gamma = (1-v^2)^{-1/2}$, $V^2 = V_\mu V^\mu = 1$, the connection between the field tensors $F_{\mu\nu}$ and $H_{\mu\nu}$ can be written as

$$\mu H_{\mu\nu} = F_{\mu\nu} + \varkappa(F_\mu V_\nu - F_\nu V_\mu), \tag{2.3}$$

where $\varkappa \equiv \varepsilon\mu - 1$, $F_\mu \equiv F_{\mu\varrho} V^\varrho$. The relation (2.3) can be brought into a compact form by making use of the following matrix

$$b_{\mu\nu} = g_{\mu\nu} + (n-1) V_\mu V_\nu, \tag{2.4}$$

where the refractive index $n = \sqrt{\varepsilon\mu}$. We first note that this matrix has the remarkable property that its p'th power is obtained by replacing n by n^p:

$$(b^p)_{\mu\nu} = g_{\mu\nu} + (n^p - 1) V_\mu V_\nu. \tag{2.5}$$

This is even true for negative integers p as well as for $p = 0$, i.e. for all integers. We can now write eq. (2.3) as

$$\mu H_{\mu\nu} = (b^2)_\mu^\varrho (b^2)_\nu^\sigma F_{\varrho\sigma}. \tag{2.6}$$

By introducing the four-potential by the relation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and exploiting the freedom in the choice of the potential components to impose the subsidiary condition in the conventional form (Lorentz gauge):

$$\Lambda^F(x) \equiv (b^2)^{\mu\nu} \partial_\mu A_\nu = \partial \cdot A + \varkappa \partial \cdot V A \cdot V = 0, \tag{2.7}$$

where $\partial \cdot A \equiv \partial_\mu A^\mu$, we can insert (2.6) into (2.2 b) to obtain

$$(b^2)^{\varrho\sigma} \partial_\varrho \partial_\sigma A_\mu = [\Box + \varkappa(\partial \cdot V)^2] A_\mu = 0 \tag{2.8}$$

$(\Box \equiv \partial_\mu \partial^\mu)$. Eq. (2.2 a) is automatically satisfied.

It can be verified that the field equations (2.8) also follow from a variational principle in which the Lagrangian density has the form

$$L = -\tfrac{1}{4} F_{\mu\nu} H^{\mu\nu} - \frac{1}{2\mu} (\Lambda^F)^2. \tag{2.9}$$

Note, however, that the variational equations obtained from this Lagrangian are equivalent to Maxwell's equations (2.2b) only when we impose the subsidiary condition $\Lambda^F = 0$.

In order to quantize the field we may apply the canonical procedure and take the expression (2.9) as the starting point. The canonically conjugate momenta π^μ are defined by

$$\pi^\mu = \frac{\partial L}{\partial \partial_0 A_\mu}, \tag{2.10}$$

which mean that in the present case

$$\pi^\mu = H^{\mu 0} - \frac{1}{\mu} (b^2)^{\mu 0} \Lambda^F. \tag{2.11}$$

In the quantum theory the field variables become operators satisfying certain commutation rules, which are constructed without regard to the subsidiary condition. It is thus clear from (2.11) and (2.9) that because of the $(\Lambda^F)^2$-term in the Lagrangian density we avoid the result $\pi^0 = 0$, as we must in order to be able to apply the canonical quantization scheme to the case $\mu = 0$. Accordingly, in quantum theory Λ^F must be considered as a non-vanishing operator. The canonical commutation rules are postulated as

$$[\pi^\mu(x), A_\nu(x')]_{x_0 = x_0'} = -i g^\mu_\nu \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{2.12a}$$

$$[\pi^\mu(x), \pi^\nu(x')]_{x_0 = x_0'} = 0, \quad [A_\mu(x), A_\nu(x')]_{x_0 = x_0'} = 0, \tag{2.12b}$$

corresponding to the interpretation of the field as a mechanical system of infinitely many degrees of freedom.

The Hamiltonian is found as

$$\left.\begin{aligned} \mathscr{H} = \int (\pi^\mu \partial_0 A_\mu - L)\, dV = \int [H^{\mu 0} \partial_0 A_\mu + \tfrac{1}{4} F_{\mu\nu} H^{\mu\nu} + \\ + \frac{1}{\mu} \Lambda^F (-(b^2)^{\mu 0} \partial_0 A_\mu + \tfrac{1}{2} \Lambda^F)]\, dV, \end{aligned}\right\} \tag{2.13}$$

where $\partial_0 A_\mu$ can be eliminated by means of (2.11), so that $\mathscr{H}$ becomes a function of $\partial_k A_\mu$, π^μ and V_μ. Now it can be shown in general (cf. for instance Källén's article(17), p. 174), that the quantal equations of motion in the Heisenberg picture are formally in agreement with eqs. (2.11) and (2.8), when the latter are looked upon as operator equations.

The relativistic invariance of the canonical quantization procedure is conveniently shown by means of the covariant commutator for the components of the four-potential at arbitrary space-time relative distances, and we shall return to this topic in the next section in connection with momentum space considerations. Let us, however, here notice the following point: The relativistic invariance of the quantization procedure for a closed system is often ascertained by verifying that the operators for linear four-momentum P_μ and angular momentum $M_{\mu\nu}$ are constants of motion, and moreover that they can be identified, respectively, with the Hermitian operator $\mathscr{P}_\mu$ generating infinitesimal translations and the Hermitian operator $\mathscr{M}_{\mu\nu}$ generating infinitesimal rotations (Lorentz transformations) in four-space. The latter operators satisfy the relations

$$i[\mathscr{P}_\mu, A_\nu(x)] = \partial_\mu A_\nu(x) \tag{2.14 a}$$

$$i[\mathscr{M}_{\mu\nu}, A^\sigma(x)] = x_\mu \partial_\nu A^\sigma(x) - x_\nu \partial_\mu A^\sigma(x) + I^{\sigma\varrho}_{\mu\nu} A_\varrho(x), \tag{2.14 b}$$

where $I^{\sigma\varrho}_{\mu\nu} = g^\sigma_\mu g^\varrho_\nu - g^\varrho_\mu g^\sigma_\nu$. Here $\mathscr{P}_\mu$ and $\mathscr{M}_{\mu\nu}$ may be time-dependent operators and should be taken at the time $t = x_0$, where x_0 is the time argument appearing in $A_\nu(x)$.

Let us now compare eqs. (2.14) with the equations one obtains by replacing $\mathscr{P}_\mu$ and $\mathscr{M}_{\mu\nu}$ by the field operators P_μ and $M_{\mu\nu}$. We have

$$P_\mu = \int S_{\mu 0}\, dV, \tag{2.15}$$

where $S_{\mu\nu}$ is the canonical energy-momentum tensor

$$S_{\mu\nu} = -g_{\mu\nu} L + \frac{\partial L}{\partial \partial^\nu A_\alpha} \partial_\mu A_\alpha. \tag{2.16}$$

The four-momentum P_μ is a constant of motion in virtue of the field equations, and by means of (2.15) and (2.16) we readily find that (2.14 a) remains valid when P_μ is present.

The study of angular momentum is somewhat more complicated; characteristic ambiguities in the formalism are shown very explicitly. By using

Noether's theorem for an infinitesimal Lorentz transformation under which we know how the field entities transform, we obtain

$$\partial^{\sigma} M_{\sigma\mu\nu} + \frac{\partial L}{\partial V^{\sigma}} I^{\sigma\varrho}_{\mu\nu} A_{\varrho} = 0, \tag{2.17}$$

$$\text{where} \quad M_{\sigma\mu\nu} = x_{\mu} S_{\nu\sigma} - x_{\nu} S_{\mu\sigma} + \frac{\partial L}{\partial \partial^{\sigma} A^{\alpha}} I^{\alpha\beta}_{\mu\nu} A_{\beta}. \tag{2.18}$$

Hence it is natural to define the angular momentum of the field as

$$M_{\mu\nu} = \int M_{0\mu\nu} \, dV, \tag{2.19}$$

since this expression is in formal agreement with the angular momentum expression for an electromagnetic field in the vacuum. It should be noted, however, that according to (2.17) (considered as an operator equation), the angular momentum defined in this way is not a constant of motion. The present feature of the theory is a direct consequence of the fact that we are considering a non-closed physical system; the Lagrangian (2.9) describes the field and its interaction with the medium but not the medium itself. Ambiguities in the formalism should therefore be expected. (In the classical theory, where one can establish the correspondence with the Maxwell field simply by putting $\Lambda^F = 0$, it can readily be shown that P_{μ} given by (2.15) and $M_{\mu\nu}$ given by (2.19) are equal respectively to the total four-momentum and the angular momentum calculated with the use of Minkowski's energy-momentum tensor $S^M_{\mu\nu}$:

$$S^M_{\mu\nu} = - F_{\mu\alpha} H_{\nu}^{\cdot\alpha} + \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} H^{\alpha\beta}.) \tag{2.20}$$

Going back to quantum theory, we can now also verify the validity of the commutator relation (2.14b), when $\mathscr{M}_{\mu\nu}$ is replaced by $M_{\mu\nu}$ determined by (2.18) and (2.19). We have thus found that the linear momentum and the angular momentum operators can be identified with the corresponding generators for infinitesimal transformations in four-space, exactly as for conventional vacuum quantum electrodynamics. It must be borne in mind, however, that the angular momentum is no longer a constant of motion.

3. Transition to Momentum Space

We now write the four-potential in the form of a four-dimensional Fourier integral

$$A_\mu(x) = (2\pi)^{-\frac{3}{2}} \int dk\delta(k^2 + \varkappa(k \cdot V)^2) e^{-ik \cdot x} A_\mu(k), \tag{3.1}$$

where $dk = dk_0 d\boldsymbol{k}$ and the delta function has been introduced because of the field equations (2.8), which lead to the condition

$$k^2 + \varkappa(k \cdot V)^2 = 0. \tag{3.2}$$

Solving this equation with respect to k_0 we obtain two solutions $k_0 = k_a$ and $k_0 = k_b$, where

$$k_{a,b} = \frac{\varkappa V_0 \boldsymbol{k} \cdot \boldsymbol{V} \pm \sqrt{(1 + \varkappa V_0^2) \boldsymbol{k}^2 - \varkappa(\boldsymbol{k} \cdot \boldsymbol{V})^2}}{1 + \varkappa V_0^2}. \tag{3.3}$$

Here k_a refers to the upper sign, which corresponds to the positive solution in the case of a vacuum field. Note that the expression (3.3) is always real, and that $k_{a,b}(\boldsymbol{k}) = - k_{b,a}(-\boldsymbol{k})$. Owing to the space-like character of the four-vector k_μ, there exists a class of inertial systems in which k_a may be negative. This is the class for which $\varkappa \boldsymbol{V}^2 > 1$, i.e. $n^2 v^2 > 1$. In such a system there is a half cone with opening angle 2α around the opposite direction of $\boldsymbol{v}$ on which $k_a = 0$; inside the cone k_a is negative and outside it is positive. The behaviour is illustrated in Fig. 1. The opening angle is determined by

$$\cos\alpha = \frac{1}{\sqrt{\varkappa}|\boldsymbol{V}|}. \tag{3.4}$$

Let us also choose the coordinate axes so that $V_1 = |\boldsymbol{V}|, V_2 = V_3 = 0$, and use eq. (3.2) to derive the following equation for a surface $k_0 = \text{const}$ in $\boldsymbol{k}$ = space:

$$\frac{[k_1 + \varkappa k_0 V_0 |\boldsymbol{V}| (1 - \varkappa \boldsymbol{V}^2)^{-1}]^2}{n^2 k_0^2 (1 - \varkappa \boldsymbol{V}^2)^{-2}} + \frac{k_2^2}{n^2 k_0^2 (1 - \varkappa \boldsymbol{V}^2)^{-1}} + \frac{k_3^2}{n^2 k_0^2 (1 - \varkappa \boldsymbol{V}^2)^{-1}} = 1. \tag{3.5}$$

when $\varkappa \boldsymbol{V}^2 < 1$ this is the equation of an ellipsoid; when $\varkappa \boldsymbol{V}^2 > 1$ it is the equation of a two-sheet hyperboloid such that the left hand sheet (the sheet extending towards $k_1 \to -\infty$) corresponds to the relation $k_0 = k_a(\boldsymbol{k})$ and the right hand sheet corresponds to the relation $k_0 = k_b(\boldsymbol{k})$. When $\varkappa \boldsymbol{V}^2 = 1$ the surface $k_0 = \text{const}$ in $\boldsymbol{k}$ –space degenerates into an elliptic paraboloid.

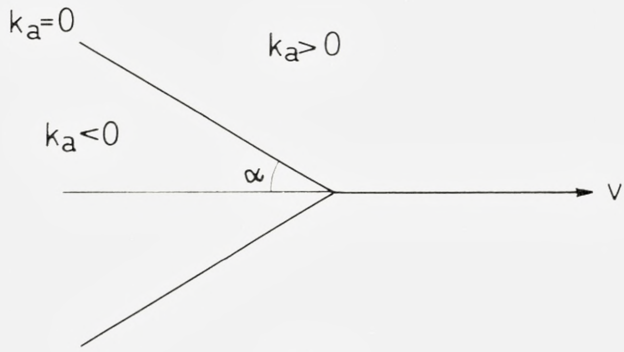

Fig. 1.

We now write the delta function in (3.1) as a sum of two terms involving the delta functions of $(k_0 - k_a)$ and $(k_0 - k_b)$, and carry out the integration over k_0. The result is conveniently written as

$$\left.\begin{aligned} &A_\mu(x) = \\ &= \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{k} \left[\frac{\mu}{(1+\varkappa V_0^2)(k_a - k_b)}\right]^{\frac{1}{2}} (b^{-1})_\mu^\nu (e^{-ik\cdot x} a_\nu(\boldsymbol{k}) + e^{ik\cdot x} a_\nu^\dagger(\boldsymbol{k})), \end{aligned}\right\} \quad (3.6)$$

where $k_0 = k_a(\boldsymbol{k}) = -k_b(-\boldsymbol{k})$. The reason for writing the expansion in the particular form (3.6) is that we thereby obtain the usual commutation rules for the components a_μ:

$$[a_\mu(\boldsymbol{k}), a_\nu^\dagger(\boldsymbol{k}')] = -g_{\mu\nu}\delta(\boldsymbol{k} - \boldsymbol{k}') \quad (3.7\,\mathrm{a})$$

$$[a_\mu(\boldsymbol{k}), a_\nu(\boldsymbol{k}')] = [a_\mu^\dagger(\boldsymbol{k}), a_\nu^\dagger(\boldsymbol{k}')] = 0, \quad (3.7\,\mathrm{b})$$

after an insertion into eqs. (2.12).

Let us now transform the commutation rules to a symmetrical four-dimensional form. We obtain from (3.6) and (3.7), when x and x' are arbitrary space-time points,

$$[A_\mu(x), A_\nu(x')] = -\frac{i\mu}{n}(b^{-2})_{\mu\nu} D^M(x - x'), \quad (3.8)$$

$$\text{where } D^M(x) = -\frac{in}{(2\pi)^3}\int dk e^{-ik\cdot x}\delta(k^2 + \varkappa(k\cdot V)^2)\varepsilon(k\cdot V) \quad (3.9)$$

and ε is the step function, $\varepsilon(t) = \pm 1$ for $t \gtrless 0$. It is apparent that $D^M(x)$ is a Lorentz invariant, so that eq. (3.8) is a covariant equation. The invariance of the quantization procedure can now be verified on the basis of eq. (3.8), by using this equation plus eq. (2.11) to recover the commutation relations (2.12). This procedure has been carried through by JAUCH and WATSON, so we need not enter into this. JAUCH and WATSON also gave an integral representation of the invariant commutator function (it should be mentioned that the expression (3.9) is equal to $-n$ times the D-function defined by these authors). It is instructive to carry out the integrations in the expression (3.9) in the inertial rest frame $\mathring{K}$ to obtain

$$D^M(x) = D^M(\mathring{x}) = -\frac{1}{2\pi}\delta\left(\mathring{\boldsymbol{x}}^2 - \frac{1}{n^2}\mathring{x}_0^2\right)\varepsilon(\mathring{x}_0), \tag{3.10}$$

where we have added a superscript zero above symbols pertaining to $\mathring{K}$. From (3.10) it is apparent that $D^M(x-x')$ is nonvanishing only when x and x' can be connected by light signals, similarly as in the case $\varkappa = 0$.

Let us now make use of the expansion (3.6) to calculate the four-momentum P_μ defined by (2.15) and (2.16). By inserting (2.9) we find after some calculation the compact expression

$$P_\mu = -\tfrac{1}{2}\int d\boldsymbol{k}\, k_\mu \{a_\nu(\boldsymbol{k}), a^{\nu\dagger}(\boldsymbol{k})\}, \tag{3.11}$$

where $P_0 = \mathscr{H}$ (cf. (2.13)) and the curly bracket denotes the anticommutator.

Instead of using (3.6), it is sometimes convenient to employ an expansion which runs over discrete values of $\boldsymbol{k}$. A discrete and covariant expansion can be constructed in the following way. We imagine that the field is enclosed within a box with linear extensions $\mathring{L}_x, \mathring{L}_y, \mathring{L}_z$ in the rest frame $\mathring{K}$ and impose the usual periodicity conditions at the walls. Hence

$$\mathring{k}_x \mathring{L}_x = 2\pi m_x, \mathring{k}_g \mathring{L}_y = 2\pi m_y, \mathring{k}_z \mathring{L}_z = 2\pi m_z, \tag{3.12}$$

where the m's are integers. The volume of the box is $\mathring{\mathscr{V}} = \mathring{L}_x \mathring{L}_y \mathring{L}_z$. It is obvious that after the transition to another inertial frame K, with respect to which $\mathring{K}$ moves with the velocity $\boldsymbol{v}$, the above periodicity conditions at the walls of the box are in general lost. Instead, each Fourier component of the field is periodic at corresponding points of the fictitious boundaries of a "box" whose volume is determined by the properties of propagation of the field component. This topic is discussed in detail elsewhere, both in a statistical treatment(18) and in an exposition upon coherence theory of black-body radiation(19). Here we recall that the plane wave component with wave vector $\mathring{\boldsymbol{k}}$ *in* $\mathring{K}$ corresponds in K to a fictitious box with the volume

$$\mathscr{V}_k = \frac{n\mathring{\mathscr{V}}}{\gamma(n + \boldsymbol{v}\cdot\mathring{\boldsymbol{k}}/|\mathring{\boldsymbol{k}}|)}, \tag{3.13}$$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$. The four-potential is expanded as

$$\left.\begin{aligned} &A_\mu(x) = \\ &= \sum_{\boldsymbol{k}} \left[\frac{\mu}{(1 + \varkappa V_0^2)(k_a - k_b)\mathscr{V}_k}\right]^{\frac{1}{2}} (b^{-1})^\nu_\mu (e^{-ik\cdot x} a_\nu(\boldsymbol{k}) + e^{ik\cdot x} a^\dagger_\nu(\boldsymbol{k})). \end{aligned}\right\} \tag{3.14}$$

By inserting the expression (3.14) into the commutation rules (2.12) we can verify that the nonvanishing commutator (3.7 a) is now changed into

$$[a_\mu(\boldsymbol{k}), a^\dagger_\nu(\boldsymbol{k}')] = -g_{\mu\nu}\delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{3.15}$$

while eqs. (3.7 b) remain unchanged. To this end we take into account that, in the limit of a large normalization volume $\mathring{\mathscr{V}}$, the sum over discrete $\boldsymbol{k}$ can be replaced by an integral over the m's such that $dm = dm_x dm_y dm_z = \partial(m)/\partial(\boldsymbol{k})d\boldsymbol{k}$, and from ref. 19 we recall that

$$dm = (2\pi)^{-3}\mathscr{V}_k d\boldsymbol{k}. \tag{3.16}$$

The Subsidary Condition

Let us now turn our attention to the subsidiary condition which, as was pointed out already in connection with eq. (2.9), must be imposed in order to maintain the connection with Maxwell's equations. First, let us consider for a moment the classical case, and notice that the Lorentz condition (2.7) leads to the following equations in Fourier space*:

$$l(\boldsymbol{k})\cdot a(\boldsymbol{k}) = 0, \quad l(\boldsymbol{k})\cdot a^*(\boldsymbol{k}) = 0, \tag{3.17 a}$$

$$\text{where} \quad l_\mu(\boldsymbol{k}) \equiv b^\nu_\mu k_\nu,\ l^2 = 0. \tag{3.17 b}$$

In conformity with usual practice let us decompose the potential by means of a $\boldsymbol{k}$ – dependent basis

$$a_\mu(\boldsymbol{k}) = \sum_{\lambda=0}^{3} e^{(\lambda)}_\mu(\boldsymbol{k})\, a^{(\lambda)}(\boldsymbol{k}), \tag{3.18}$$

so that

$$g^{\mu\nu} e^{(\lambda)}_\mu e^{(\lambda')}_\nu = g^{\lambda\lambda'}, \quad \sum_{\lambda,\lambda'=0}^{3} g_{\lambda\lambda'} e^{(\lambda)}_\mu e^{(\lambda')}_\nu = g_{\mu\nu}. \tag{3.19}$$

* The star denotes complex conjugate.

Let us choose the components $e_{\mu}^{(\lambda)}$ in the following form

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} &= \frac{\boldsymbol{l}}{l_0}, \quad \boldsymbol{e}^{(2)} = \frac{(\boldsymbol{V}\times\boldsymbol{l})\times\boldsymbol{l}}{|(\boldsymbol{V}\times\boldsymbol{l})\times\boldsymbol{l}|} \\ \boldsymbol{e}^{(3)} &= \frac{\boldsymbol{V}\times\boldsymbol{l}}{|\boldsymbol{V}\times\boldsymbol{l}|}, \quad \boldsymbol{e}^{(0)} = 0, \; e_0^{(\lambda)} = \delta_{\lambda 0}. \end{aligned}\right\} \tag{3.20}$$

Thereby eqs. (3.19) are satisfied, and the classical Lorentz condition (3.17) takes on the simple form

$$a^{(1)} - a^{(0)} = 0, \quad a^{(1)*} - a^{(0)*} = 0. \tag{3.21}$$

In quantum theory the subsidiary condition cannot be taken over as a set of operator equations corresponding to eqs. (3.21) since this would lead to the relation

$$[a^{(1)}, a^{(1)\dagger}] = [a^{(0)}, a^{(0)\dagger}], \tag{3.22}$$

which contradicts the relation $[a^{(\lambda)}, a^{(\lambda')\dagger}] = -g^{\lambda\lambda'}$ following from (3.15). This is evidently a consequence of the fact that we have quantized the field according to the canonical procedure on the basis of the Lagrangian (2.9), without regard to the Lorentz condition. Nor should (3.21) be postulated valid when acting on a state vector $|\Psi\rangle$ (Fermi's method), since (3.22) would then remain as a relation between the expectation values. Instead, we shall employ the method due to S. Gupta and K. Bleuler (cf., for instance, ref. 17) with the modifications that are necessary because of the presence of the medium.

Just as in the case $\varkappa = 0$ we introduce a Hermitic and unitary metric operator η:

$$\eta = \eta^{\dagger} = \eta^{-1}, \tag{3.23}$$

and define the expectation value of an operator F as $\bar{F} = \langle\Psi|\eta F|\Psi\rangle$. The metric operator is further required to satisfy the usual commutation and anticommutation relations

$$\left.\begin{aligned} [\eta, a^{(\lambda)}(\boldsymbol{k})] &= 0 \quad (\lambda = 1,2,3) \\ \{\eta, a^{(0)}(\boldsymbol{k})\} &= 0. \end{aligned}\right\} \tag{3.24}$$

Both in the commutator (3.15) and in the expansion (3.14) $a_{\nu}^{\dagger}(\boldsymbol{k})$ is replaced by $\tilde{a}_{\nu}(\boldsymbol{k})$, where

$$\tilde{a}_{\nu}(\boldsymbol{k}) \equiv \eta a_{\nu}^{\dagger}(\boldsymbol{k})\eta. \tag{3.25}$$

By decomposing into the polarization directions, the commutation relations can then be written as

$$[a^{(\lambda)}(\boldsymbol{k}), a^{(\lambda')\dagger}(\boldsymbol{k}')] = \delta_{\lambda\lambda'}\delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{3.26}$$

and $a^{(\lambda)}$ and $a^{(\lambda)\dagger}$ can be interpreted as annihilation and creation operators for the "phenomenological" photons. The potential components $A_\mu(x)$ become complex quantities; the simple Hermitic property of $A_i(x)$ and the anti-Hermitic property of $A_0(x)$ which hold in the case $\varkappa = 0$ are in general lost. However, the expectation values of the potential components must satisfy the same reality conditions as the classical fields, i.e. $\langle\Psi|\eta A_\mu(x)|\Psi\rangle = \langle\Psi|A_\mu^\dagger(x)\eta|\Psi\rangle$, or

$$A_\mu(x) = \eta A_\mu^\dagger(x)\eta. \tag{3.27}$$

By means of (3.24) it can actually be verified that the relation (3.27) is satisfied, so that the reality properties are correct. The subsidiary condition is written as

$$(a^{(1)} - a^{(0)})|\Psi\rangle = 0, \tag{3.28}$$

which involves absorption operators only. Hence the conflict (3.22) is avoided, yet (3.28) is sufficient to yield the relation

$$\langle\Psi|\eta\Lambda^F|\Psi\rangle = 0, \tag{3.29}$$

which expresses the correspondence with the classical Lorentz condition. When the expectation values are taken, the field equations (2.8) are thus equivalent to Maxwell's equations. Similarly we find that $\langle\Psi|\eta\Lambda^{F2}|\Psi\rangle = 0$, and that the influence of the Λ^F-dependent term in the Lagrangian (2.9) on the field-dependent part of P_μ vanishes, so that the correspondence with the classical field is established also for the conserved quantities P_μ. In fact, by omitting the zero point contributions we can write P_μ as

$$P_\mu = \sum_{\boldsymbol{k}} k_\mu \sum_{\lambda=0}^{3} a^{(\lambda)\dagger} a^{(\lambda)} \tag{3.30 a}$$

and $$\langle\Psi|\eta P_\mu|\Psi\rangle = \sum_{\boldsymbol{k}} k_\mu (N^{(2)} + N^{(3)}), \tag{3.30 b}$$

where $N^{(2)}$, $N^{(3)}$ mean the numbers of photons polarized in the transversal directions. It should be noted that, as a characteristic difference from the case $\varkappa = 0$, the phrase "transversal" refers to the directions which are transversal to the vector $\boldsymbol{l} = \boldsymbol{k} + (n-1)\boldsymbol{V}k \cdot V$.

From (3.30) it follows that the energy of a photon is equal to $k_0 = k_a$ which, according to the remarks in the beginning of this section, may be negative in the inertial systems for which $\varkappa \boldsymbol{V}^2 > 1$. The space-like character of the four-momentum k_μ is a direct consequence of the fact that Minkowski's momentum density in the rest frame is put equal to $(\mathring{\boldsymbol{D}} \times \mathring{\boldsymbol{B}})$, and hence includes also a contribution from the moving constituent particles of the medium. The classical aspects of this subject have been treated elsewhere[(4)].

We may write an arbitrary state vector $|\Psi\rangle$ as

$$|\Psi\rangle = |\Psi_T\rangle \prod_{\boldsymbol{k}} |\Phi_{\boldsymbol{k}}\rangle, \tag{3.31}$$

where $|\Psi_T\rangle$ contains transversal photons only and $\Phi_{\boldsymbol{k}}$ contains a mixture of longitudinal and scalar photons which is arbitrary apart from the fact that it must be compatible with the condition (3.28). A certain mixture corresponds to a certain value for the gauge and has no influence on the physical quantities. The argument proceeds similarly as for an electromagnetic field in vacuum[(17)], so we abstain from a detailed exposition.

As regards the Lorentz invariance of the theory it should first be noted that by choosing the expansion of the four-potential in the form (3.14) we obtain a convenient relativistic behaviour of the Fourier components. For by observing that $[(1 + \varkappa V_0^2)(k_a - k_b)]^{-1} d\boldsymbol{k}$ is a Lorentz invariant and that dm given by (3.16) also is a Lorentz invariant so that $\mathscr{V}_k d\boldsymbol{k} = \mathring{\mathscr{V}} d\mathring{\boldsymbol{k}}$ (cf. ref. 19), we see that

$$\frac{\mu}{(1 + \varkappa V_0^2)(k_a - k_b)\mathscr{V}_k} = \frac{\mu}{2n|\mathring{\boldsymbol{k}}|\mathring{\mathscr{V}}} = \text{invariant}. \tag{3.32}$$

Thus the expectation value of the Fourier component $a_\mu(\boldsymbol{k})$ will simply transform like a four-vector. Consequently, the Lorentz invariance of the total photon number $N(\boldsymbol{k})$ corresponding to some wave vector $\boldsymbol{k}$ (i.e. the expectation value of $-g^{\mu\nu}\tilde{a}_\mu a_\nu = \sum_{\lambda=0}^{3} a^{(\lambda)\dagger} a^{(\lambda)}$) follows in a very natural way.

Remarks

We finish this section with a few remarks related to earlier works on the subject. The only works we are aware of in which the method of decomposing the potential by means of the unit vectors (3.20) has been employed, are the two papers by C. Muzikář[(11)]. He developed a covariant theory based on the Coulomb gauge in the rest frame and found, similarly as above, that the physically important polarization directions are the directions transverse to $\boldsymbol{l}$. . On the other hand, the result (3.30 b) is in disagreement with the result obtained by JAUCH and WATSON[6]; they found that the physically important polarization directions were the directions transverse to the vector $\boldsymbol{k}$. It can be checked that (for $\varkappa, V \neq 0$) the unit vector $\boldsymbol{e}^{(1)}$ defined by (3.20) is equal to $\boldsymbol{k}/|\boldsymbol{k}|$ when, and only when, $\boldsymbol{k}$ is parallel or antiparallel to $\boldsymbol{V}$. We think that it is desirable to go into some detail and give the reason for this rather essential discrepancy. Thus, by transforming the formalism to our notation, let us introduce a new basis $\varepsilon_\mu^{(\lambda)}(\boldsymbol{k})$ where the differences from (3.20) are contained in the following terms:

$$\boldsymbol{\varepsilon}^{(1)} = \boldsymbol{k}/|\boldsymbol{k}|, \boldsymbol{\varepsilon}^{(2)} = \frac{(\boldsymbol{V} \times \boldsymbol{k}) \times \boldsymbol{k}}{|(\boldsymbol{V} \times \boldsymbol{k}) \times \boldsymbol{k}|}, \boldsymbol{\varepsilon}^{(3)} = \frac{\boldsymbol{V} \times \boldsymbol{k}}{|\boldsymbol{V} \times \boldsymbol{k}|}. \tag{3.33}$$

These unit vectors were essentially chosen by JAUCH and WATSON. Further, the Fourier expansions (of the Schrödinger operators) chosen by these authors are equivalent to the following Fourier expansions of the Heisenberg operators:

$$A_\mu = (2\pi)^{-\frac{3}{2}} \int d\boldsymbol{k} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)} \alpha^{(\lambda)} (e^{-ik\cdot x} c^{(\lambda)} + e^{ik\cdot x} c^{(\lambda)\dagger}) \tag{3.34 a}$$

$$\pi_\mu = i(2\pi)^{-\frac{3}{2}} \int d\boldsymbol{k} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)} \beta^{(\lambda)} (e^{-ik\cdot x} c^{(\lambda)} - e^{ik\cdot x} c^{(\lambda)\dagger}), \tag{3.34 b}$$

where $\alpha^{(\lambda)}$ and $\beta^{(\lambda)}$ are numerical factors. The expansions (3.34) are inserted into the Hamiltonian in its reduced form after the subsidiary condition has been imposed, so that only physically important terms are left. In this way the authors find that only the polarization directions $\lambda = 2$ and $\lambda = 3$ (based on (3.33)) remain, and that the Hamiltonian takes the desired form involving $k_a \sum_{\lambda=2}^{3} \{c^{(\lambda)}, c^{(\lambda)\dagger}\}$ if the pertinent factors $\alpha^{(\lambda)}, \beta^{(\lambda)}$ in (3.34) are assigned the following values

$$\left.\begin{aligned} \alpha^{(2)} &= \frac{1}{2n\,|\boldsymbol{k}|}[\mu(1+\varkappa V_0^2)(k_a - k_b)]^{\frac{1}{2}}, \alpha^{(3)} = \sqrt{\mu}[(1+\varkappa V_0^2)(k_a - k_b)]^{-\frac{1}{2}} \\ \beta^{(2)} &= \frac{1}{2\,\alpha^{(2)}}, \quad \beta^{(3)} = \frac{1}{2\,\alpha^{(3)}}. \end{aligned}\right\} \tag{3.35}$$

By letting $\beta^{(\lambda)} = 1/(2\alpha^{(\lambda)})$ also for $\lambda = 0$ and $\lambda = 1$ one finds that the commutation rules (2.12) are satisfied if $[c^{(\lambda)}(\boldsymbol{k}), c^{(\lambda')\dagger}(\boldsymbol{k}')] = -g^{\lambda\lambda'}\delta(\boldsymbol{k}-\boldsymbol{k}')$.

It should be pointed out, however, that the expansions (3.34) are in-compatible with the definition equation (2.11) for the canonical momentum. In fact, if we make use of (3.34a) and (2.11) we can write

$$\left.\begin{aligned} \pi_\mu = & -\frac{i}{\mu}(2\pi)^{-3/2}[(b^2)^\varrho_\mu (b^2)^\sigma_0 - (b^2)^\sigma_\mu (b^2)^\varrho_0 - (b^2)_{\mu 0}(b^2)^{\varrho\sigma}]\cdot \\ & \cdot \int d\boldsymbol{k}\, k_\varrho \sum_{\lambda=0}^{3} \varepsilon^{(\lambda)}_\sigma a^{(\lambda)} (e^{-ik\cdot x} c^{(\lambda)} - e^{ik\cdot x} c^{(\lambda)\dagger}), \end{aligned}\right\} \quad (3.36)$$

which shows a more complicated behaviour than Jauch and Watson's expression (3.34b). If we consider the various polarization directions separately, we find however that the contributions to (3.34b) and (3.36) from the direction $\lambda = 3$ are equal for any $\boldsymbol{k}$. For $\lambda = 2$ we find a disagreement, except in the special cases $\boldsymbol{k}\times\boldsymbol{V} = 0$ or $\varkappa = 0$. In the limiting cases $\boldsymbol{k}\times\boldsymbol{V} \to 0$ or $\varkappa \to 0$ the unit vectors (3.33) tend to coincide with the unit vectors (3.20).

The next point that we shall consider, is the possibility of using instead of (2.9) the following expression for the Lagrangian density:

$$L_1 = -\frac{1}{2\mu}(\partial_\nu A_\mu)\partial^\nu A^\mu - \frac{\varkappa}{2\mu}(V\cdot\partial A_\mu)(V\cdot\partial A^\mu). \quad (3.37)$$

(It is well known that in the vacuum case $\varkappa = 0$ either of the expressions (3.37) and (2.9) can be used.) Actually, R. Dobbertin (12) has worked out a covariant theory on the basis of (3.37). The theory so constructed appears to have some attractive, simple properties: The variational equations obtained from (3.37) are the same as our previous (2.8) or (3.2); moreover, one can employ the Fourier expansion (3.6), simplified in the sense that $(b^{-1})^\nu_\mu$ is replaced by g^ν_μ, in order to obtain an expression for the total four-momentum in the form (3.30a). The photon four-momentum thus also becomes equal to k_μ.

However, the following essential feature of the theory should be pointed out: The canonical four-momentum calculated from (3.37) does *not* correspond to the classical canonical four-momentum $P_\mu^{\mathrm{Maxw.}}$ of the Maxwell field calculated from the Lagrangian density

$$L^{\mathrm{Maxw.}} = -\tfrac{1}{4}F_{\mu\nu}H^{\mu\nu}, \quad (3.38)$$

even if the comparison is made after the application of the subsidiary condition. Let us consider the Hamiltonian as an example: By using (3.37) and (3.38) to establish the relation

$$\left.\begin{aligned} L^{\text{Maxw.}} - L_1 = & -\frac{\varkappa}{2\mu}(\partial_\nu V\cdot A)\partial^\nu V\cdot A - \frac{\varkappa^2}{2\mu}(V\cdot\partial V\cdot A)^2 + \frac{1}{2\mu}(\Lambda^F)^2 + \\ & + \partial^\mu\left[\frac{\varkappa}{\mu}V\cdot A(V\cdot\partial A_\mu - V_\mu\partial\cdot A) + \frac{1}{2\mu}(A\cdot\partial A_\mu - A_\mu\partial\cdot A)\right] \end{aligned}\right\} \quad (3.39)$$

and disregarding the divergence term, we find

$$\left.\begin{aligned} \mathscr{H}^{\text{Maxw.}} - \mathscr{H}_1 = & \frac{1}{\mu}\int\left[-\tfrac{1}{2}(\Lambda^F)^2 + \Lambda^F(b^2)^{0\mu}\partial_0 A_\mu - \right. \\ & \left. - \varkappa(b^2)^{0\mu}(\partial_0 V\cdot A)(\partial_\mu V\cdot A) + \frac{\varkappa}{2}(b^2)^{\mu\nu}(\partial_\mu V\cdot A)(\partial_\nu V\cdot A)\right]dV. \end{aligned}\right\} \quad (3.40)$$

This relation shows that $\mathscr{H}^{\text{Maxw.}}$ and $\mathscr{H}_1$ are in general not equivalent, even when the terms containing Λ^F are disregarded. This behaviour is restricted to the case where $\varkappa \neq 0$; in a vacuum field it follows that $\mathscr{H}^{\text{Maxw.}}$ and $\mathscr{H}_1$ are equivalent.

The above feature represents a serious restriction on the applicability of the Lagrangian density L_1 in the description of the electromagnetic field. One should rather employ a Lagrangian, as for instance L given by (2.9), which maintains the connection with the Maxwell field in virtue of the subsidiary condition.

4. A Transformation Procedure applied to the Classical Field

Up till now we have considered the simple case of a pure radiation field from a conventional point of view in the sense that all the analysis has been carried out in the actual physical situation, i.e. in the presence of a uniformly moving transparent medium. Instead of going on using this method, we shall in the following present another, rather unconventional, method to construct the phenomenological theory, namely, to start from the well known expressions in the vacuum field case and then obtain the corresponding expressions in the medium field case simply by a transformation procedure. In this section we shall restrict ourselves to the classical case.

We first show how the Maxwell equations in matter, in the presence of external charges,

$$\partial^\nu H_{\mu\nu} = -j_\mu, \quad (4.1)$$

where $j_\mu = (\varrho, \boldsymbol{j})$, can be mapped into the vacuum field equations. The transformation is accomplished in two steps. First we define the B-potentials

$$B_\mu(x) = b_{\mu\nu} A^\nu(x) \tag{4.2}$$

and the differential operator

$$D_\mu = b_{\mu\nu} \partial^\nu . \tag{4.3}$$

The "field strengths" corresponding to the B-potentials are

$$G_{\mu\nu}(x) = b_\mu^\varrho b_\nu^\sigma F_{\varrho\sigma}(x) = D_\mu B_\nu(x) - D_\nu B_\mu(x). \tag{4.4}$$

The Maxwell equations (4.1) can then be written

$$D^\nu G_{\mu\nu} = -D^2 B_\mu + D_\mu D_\nu B^\nu = -J_\mu, \tag{4.5}$$

where $J_\mu = \mu (b^{-1})_\mu^\nu j_\nu$ is the "current" of the B-field satisfying the "continuity equation"

$$D^\mu J_\mu = 0. \tag{4.6}$$

The resemblance of eqs. (4.4–6) with the equations of the Maxwell field in vacuum is striking. Gauge invariance, which can be expressed as the invariance of the equations of motion under the transformation

$$A_\mu \to A_\mu + \partial_\mu \chi, \tag{4.7}$$

is in terms of the B-field expressible as the invariance under the transformation

$$B_\mu \to B_\mu + D_\mu \chi, \tag{4.8}$$

and it immediately follows that a natural choice of gauge condition corresponding to the Lorentz condition of vacuum electrodynamics is

$$D^\mu B_\mu = \partial \cdot A + \varkappa V \cdot \partial V \cdot A = 0. \tag{4.9}$$

This is the same gauge condition as our previous eq. (2.7) and it implies, as we have seen, that the Maxwell equations take on the form (2.8).

We are now in a position to complete the transformation to the Maxwell equations in vacuum by defining the vacuum field

$$A_\mu^{\text{vac}}(x) = \varrho B_\mu(y) = \varrho b_{\mu\nu} A^\nu(y), \tag{4.10}$$

$$\text{where} \quad y_\mu = b_{\mu\nu} x^\nu \tag{4.11}$$

$$\text{and} \quad \varrho = (n/\mu)^{\frac{1}{2}}. \tag{4.12}$$

It will be clear later on that the factor ϱ in (4.10) is necessary in order that the vacuum field shall acquire the correct commutation relations of a free field (cf. also the momentum space considerations below). A consequence of the definition (4.11) is that D^y_μ can be replaced by ∂^x_μ, and so the equations of motion for $A^{\text{vac}}_\mu(x)$ can be written as

$$-\Box A^{\text{vac}}_\mu(x) + \partial_\mu \partial^\nu A^{\text{vac}}_\nu(x) = -j^{\text{vac}}_\mu(x), \tag{4.13}$$

where the "vacuum current" is given by

$$j^{\text{vac}}_\mu(x) = \varrho J_\mu(y) = \varrho\mu(b^{-1})^\nu_\mu j_\nu(y). \tag{4.14}$$

This current also satisfies the continuity equation

$$\partial^\mu j^{\text{vac}}_\mu(x) = 0. \tag{4.15}$$

These considerations show that there is an intimate connection between the equations of motion of the Maxwell field in isotropic matter and in the vacuum; the Maxwell field in matter can be mapped uniquely into the Maxwell field in the vacuum. This feature is not so surprising as it might seem at first. For if we consider the equations of motion in the rest frame they will essentially consist of the differential operator $\Delta - n^2\partial^2/\partial t^2$ acting on the potentials. If we then redefine time by putting $t' = \frac{1}{n} t$ this operator will be transformed into the usual d'Alembertian operator and the equations of motion will look like the free equations of motion. What we have done in this section is just to carry out such a procedure in a manifestly covariant way.

As an example of the application of the developed formalism it is instructive to go into momentum space considerations. We may expand the vacuum field in a form similar to (3.14):

$$A^{\text{vac}}_\mu(x) = \sum_{\boldsymbol{l}} \frac{1}{[2\, l_0 \mathscr{V}_l]^{1/2}} (e^{-il\cdot x} e_\mu(\boldsymbol{l}) + e^{il\cdot x} e^*_\mu(\boldsymbol{l})), \tag{4.16}$$

where l_μ is the four-momentum of a vacuum "photon" and the normalization volume $\mathscr{V}_l$ transforms according to eq. (3.13) (with $n = 1$). With the use of (4.10) and (4.11) it follows that

$$k_\mu = (b^{-1})^\nu_\mu l_\nu. \tag{4.17}$$

In view of the invariance property (3.32) the normalization volumes $\mathring{\mathscr{V}}$ in the expansions (3.14) and (4.16) can be put equal, whereby

$$a_\mu(\boldsymbol{k}) = e_\mu(\boldsymbol{l}). \tag{4.18}$$

Note that the factor ϱ appears automatically in the transformation formula. The equations of motion for a radiation field imply that $l^2 = 0$, whereas the Lorentz condition implies that $l \cdot e = 0$. If the volume $\mathscr{V}_l$ contains only one "photon", the vector e_μ in (4.16) fulfils the normalization condition

$$e_\mu^* e^\mu = -1. \tag{4.19}$$

It is however convenient to change the normalization conditions in such a way that the "one-photon" potential of a vacuum photon (with four-momentum l_μ) takes the form

$$A_\mu^{\text{vac}}(x) = e^{-il\cdot x} e_\mu(\boldsymbol{l}) + e^{il\cdot x} e_\mu^*(\boldsymbol{l}). \tag{4.20}$$

Here e_μ still fulfils eq. (4.19), but the volume corresponding to one "photon" is now not $\mathscr{V}_l$ but instead v_l, where $v_l = 1/(2l_0)$.

The corresponding equation for one "photon" in the medium is

$$A_\mu(x) = \frac{1}{\varrho}(e^{-ik\cdot x} f_\mu + e^{ik\cdot x} f_\mu^*), \tag{4.21 a}$$

$$f_\mu = (b^{-1})_\mu^\nu e_\nu, \tag{4.21 b}$$

and the "one-photon" volume is now v_k, where

$$v_k = \frac{n}{(1 + \varkappa V_0^2)(k_a - k_b)}, \tag{4.22}$$

cf. eq. (3.32). The vacuum field equations $l^2 = 0$, $l \cdot e = 0$ and the condition (4.19) then lead to the equations

$$k^\mu k^\nu (b^2)_{\mu\nu} = k^2 + \varkappa (k \cdot V)^2 = 0 \tag{4.23 a}$$

$$k^\mu f^\nu (b^2)_{\mu\nu} = k \cdot f + \varkappa (k \cdot V)(f \cdot V) = 0 \tag{4.23 b}$$

$$f^{\mu *} f^\nu (b^2)_{\mu\nu} = f^* \cdot f + \varkappa (f^* \cdot V)(f \cdot V) = -1. \tag{4.23 c}$$

Gauge Considerations

It is instructive to make use of the above transformation method in a study of the gauge condition. We first observe that the gauge invariance of the vacuum field, which means e_μ and $(e_\mu + cl_\mu)$ are equivalent polarization vectors, implies that in the case of the medium field eqs. (4.23) are invariant under the transformation $f_\mu \to f_\mu + ck_\mu$. Normally one imposes one extra gauge condition on the polarization vector in the vacuum, namely the transversality condition $e_0 = 0$. This result can always be obtained by a gauge transformation since l_0 is different from zero. However, the corresponding condition $f_0 = 0$ in matter is not always possible, because k_0 can be zero in certain inertial frames due to the space-like character of k_μ (cf. eq. (3.4)). In fact, in the zero frequency case $k_0 = 0$ it is possible to show that one *cannot* find two independent solutions of f_μ, both with $f_0 = 0$*. *Hence the requirement $f_0 = 0$ is in general not a legitimate gauge condition.*

In order to obtain a general description we must find another gauge condition than $f_0 = 0$. Let us examine the possible choice $e_0 = 0$. By means of (4.21b) and (4.17) we then see that

$$f_0 + (n-1)V_0 f \cdot V = 0 \tag{4.24}$$

and

$$\boldsymbol{l} \cdot \boldsymbol{e} = (\boldsymbol{k} + (n-1)\boldsymbol{V} k \cdot V) \cdot \boldsymbol{e} = 0. \tag{4.25}$$

It is clear that as two independent solutions of the latter equation we can choose the vectors $\boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(3)}$ given by (3.20). Notice that hereby the condition (4.19) will be satisfied. The corresponding f's can be found from (4.21b). We have thus obtained the interesting result that our earlier choice (3.20) of unit vectors stands in intimate connection with the natural solutions of the equations of the "one-photon" problem in the transformed vacuum field case.

It is however apparent that the above gauge condition suffers from the usual drawback of vacuum electrodynamis that it is non-covariant. As we shall see now it is possible to improve this feature and make the whole theory covariant, by making use of the transformation procedure and the fact that the rest frame of the medium represents a distinguished system of reference. In this frame the frequency is always different from zero so that we can put $\overset{\circ}{f}_0 = 0$. This condition can be expressed in covariant fashion as

$$f \cdot V = 0, \tag{4.26}$$

and we shall seek to make the whole theory covariant by introducing new covariant polarization vectors which are in conformity with (4.26). First

* Assume $k_0 = 0, f_0 = 0$. Then from (4.23a, b) we get $\varkappa(\boldsymbol{k} \cdot \boldsymbol{V})^2 = \boldsymbol{k}^2, \boldsymbol{k} \cdot \boldsymbol{f} = \varkappa \boldsymbol{k} \cdot \boldsymbol{V} \boldsymbol{f} \cdot \boldsymbol{V}$. Combining these we obtain $\boldsymbol{k} \cdot \boldsymbol{f} \boldsymbol{k} \cdot \boldsymbol{V} = \boldsymbol{k}^2 \boldsymbol{f} \cdot \boldsymbol{V}$, which means that $\boldsymbol{f}$ lies in the plane determined by $\boldsymbol{k}$ and $\boldsymbol{k} \times \boldsymbol{V}$, i.e. $\boldsymbol{f} = c_1 \boldsymbol{k} \times \boldsymbol{V} + c_2 \boldsymbol{k}$. The solution parallel to $\boldsymbol{k}$ must however be excluded since it runs into conflict with eq. (4.23c) in the directions determined by $k_0 = 0$. There remains the solution $\boldsymbol{f} = c_1 \boldsymbol{k} \times \boldsymbol{V}$, i.e. there is only one polarization vector in this plane.

it should be noted that the condition (4.26) can always be obtained by a suitable gauge transformation $f_\mu \to f_\mu + ck_\mu$, since $k \cdot V$ is always different from zero. It can readily be verified that for polarization vectors f_μ satisfying (4.26) we also have $f_\mu = e_\mu$.

Now let $e_\mu^{(2)}$ be an arbitrary polarization four-vector satisfying

$$V \cdot e^{(2)} = l \cdot e^{(2)} = 0 \tag{4.27}$$

$$e^{(2)*} \cdot e^{(2)} = -1. \tag{4.28}$$

Then we can define the vector*

$$e_\mu^{(3)} = -\frac{\varepsilon_{\mu\nu\varrho\sigma} l^\nu V^\varrho (e^{(2)*})^\sigma}{l \cdot V}. \tag{4.29}$$

This vector also satisfies

$$V \cdot e^{(3)} = l \cdot e^{(3)} = 0 \tag{4.30}$$

$$e^{(3)*} \cdot e^{(3)} = -1 \tag{4.31}$$

and furthermore

$$e^{(2)*} \cdot e^{(3)} = 0. \tag{4.32}$$

Another important property is

$$\varepsilon_{\mu\nu\varrho\sigma}(e^{(2)})^\varrho (e^{(3)})^\sigma = -\frac{l_\mu V_\nu - l_\nu V_\mu}{l \cdot V}, \tag{4.33}$$

which in the rest system implies

$$\mathring{\boldsymbol{e}}^{(2)} \times \mathring{\boldsymbol{e}}^{(3)} = \mathring{\boldsymbol{l}}/|\mathring{\boldsymbol{l}}| = \mathring{\boldsymbol{k}}/|\mathring{\boldsymbol{k}}|. \tag{4.34}$$

For the following it is convenient to introduce two four-vectors more. These can however not be interpreted as polarization vectors for plane waves. Let us define

$$e_\mu^{(0)} = V_\mu, \quad e^{(0)} \cdot e^{(0)} = 1 \tag{4.35}$$

$$e_\mu^{(1)} = \frac{l_\mu - V_\mu l \cdot V}{l \cdot V}, \quad e^{(1)} \cdot e^{(1)} = -1. \tag{4.36}$$

Then we can verify that our new covariant basic vectors $e_\mu^{(\lambda)}$, $\lambda = 0, 1, 2, 3$, satisfy eqs. (3.19) (with one of the vectors replaced by its complex conjugate). It is appropriate at this place to recall that the $e_\mu^{(\lambda)}$ defined by (3.20) are *not* four-vectors.

* $\varepsilon_{0123} = 1$.

Thus, according to this new description in which both the field equations and the gauge condition are covariant, it follows that $e_\mu^{(2)}$ and $e_\mu^{(3)}$ can be taken as the two independent physical solutions describing the "one-photon" situation in the vacuum field case. The corresponding solution in the medium field case is obtained by means of eq. (4.21 b).

Let us finally make use of eq. (3.19) to calculate the polarization sum

$$\sum_{\lambda=2,3} e_\mu^{(\lambda)} e_\nu^{(\lambda)*} = -g_{\mu\nu} - \frac{l_\mu l_\nu}{(l\cdot V)^2} + \frac{l_\mu V_\nu + l_\nu V_\mu}{l\cdot V}, \tag{4.37}$$

which can be expressed in terms of f and k:

$$\sum_{\lambda=2,3} f_\mu^{(\lambda)} f_\nu^{(\lambda)*} = -(b^{-2})_{\mu\nu} - \frac{k_\mu k_\nu}{n^2(k\cdot V)^2} + \frac{k_\mu V_\nu + k_\nu V_\mu}{n^2 k\cdot V}. \tag{4.38}$$

5. Quantization of the Electromagnetic Field by Means of the Transformation Method. Use of Covariant Gauges in Configuration Space

In this section we shall make use of the transformation method introduced in the previous section and shall construct the quantum theory of the electromagnetic field in the medium by starting from the expressions of vacuum quantum electrodynamics. Besides, we shall permit the presence of general covariant gauges in the construction of the commutation rules in configuration space. (We recall that in the sections 2 and 3 a single gauge was permitted, namely the Fermi gauge.) The family of covariant gauges that we shall assume has been studied (in the case $\varkappa = 0$) by one of the authors elsewhere [(20)]. It corresponds to the Lagrangian density

$$L^{\text{vac}}(x) = -\tfrac{1}{4} F_{\mu\nu}^{\text{vac}} F^{\mu\nu\,\text{vac}} - \Lambda^{\text{vac}} \partial^\mu A_\mu^{\text{vac}} + \frac{a}{2}(\Lambda^{\text{vac}})^2, \tag{5.1}$$

where Λ^{vac} is a Lagrange multiplier field which is introduced to take care of the gauge condition, and where a is the gauge parameter. The Fermi gauge is obtained by putting $a = 1$. The field equations are obtained from eq. (5.1) as

$$-\Box A_\mu^{\text{vac}} + \partial_\mu \partial^\nu A_\nu^{\text{vac}} = \partial_\mu \Lambda^{\text{vac}}, \tag{5.2}$$

while the gauge condition emerges after variation with respect to Λ^{vac}:

$$\partial^{\mu} A_{\mu}^{\mathrm{vac}} = a\Lambda^{\mathrm{vac}}. \tag{5.3}$$

The commutation rules become

$$[A_{\mu}^{\mathrm{vac}}(x), A_{\nu}^{\mathrm{vac}}(x')] = -ig_{\mu\nu}D(x-x') - i(1-a)\partial_{\mu}\partial_{\nu}E(x-x'), \tag{5.4}$$

where $D(x)$ is the well known singular function

$$D(x) = -\frac{i}{(2\pi)^3}\int dl\varepsilon(l)\delta(l^2)e^{-il\cdot x} = -\frac{\varepsilon(x)}{2\pi}\delta(x^2) \tag{5.5}$$

and [20]

$$E(x) = \frac{i}{(2\pi)^3}\int dl\,\varepsilon(l)\delta'(l^2)e^{-il\cdot x} = \frac{\varepsilon(x)}{8\pi}\theta(x^2), \tag{5.6}$$

where the delta function is differentiated with respect to its argument.

Let us now use eqs. (4.10–12) to define the fields $B_{\mu}(y)$ and $A_{\mu}(y)$ from the vacuum field $A_{\mu}^{\mathrm{vac}}(x)$. We obtain immediately from (5.2) and (5.3) the equations of motion and gauge-condition for the B-field

$$-D^2 B_{\mu}(y) + D_{\mu}D^{\nu}B_{\nu}(y) = D_{\mu}\Lambda(y) \tag{5.7}$$

$$D^{\mu}B_{\mu}(y) = a\Lambda(y), \tag{5.8}$$

where $\Lambda(y) = \varrho^{-1}\Lambda^{\mathrm{vac}}(x)$. In the Fermi gauge ($a = 1$) eq. (5.7) reduces to the well known equations of motion. The commutation relations for the B-field become

$$\left.\begin{aligned}&[B_{\mu}(y), B_{\nu}(y')] = \\ &= \frac{1}{\varrho^2}\{-ig_{\mu\nu}D(b^{-1}(y-y')) - i(1-a)D_{\mu}^{y}D_{\nu}^{y}E(b^{-1}(y-y'))\},\end{aligned}\right\} \tag{5.9}$$

where $(b^{-1}(y-y'))_{\mu} = (x-x')_{\mu}$. At last we obtain the commutation relations for the A-field

$$\left.\begin{aligned}&[A_{\mu}(y), A_{\nu}(y')] = \\ &= \frac{1}{\varrho^2}\{-i(b^{-2})_{\mu\nu}D(b^{-1}(y-y')) - i(1-a)\partial_{\mu}^{y}\partial_{\nu}^{y}E(b^{-1}(y-y'))\}.\end{aligned}\right\} \tag{5.10}$$

In order to cast (5.10) into a more conventional form we apply the transformation (4.17) to the expression (5.5), whereby we get

$$D(b^{-1}y) = -\frac{in}{(2\pi)^3}\int dk\varepsilon(k\cdot V)\delta(k^2 + \varkappa(k\cdot V)^2)e^{-ik\cdot y}. \tag{5.11}$$

Here we have used the connections $\varepsilon(l) = \varepsilon(k \cdot V)$, $\det\{b_\mu^\nu\} = n$. Similarly

$$E(b^{-1}y) = \frac{in}{(2\pi)^3}\int dk\varepsilon(k\cdot V)\delta'(k^2 + \varkappa(k\cdot V)^2)e^{-ik\cdot y}. \tag{5.12}$$

Putting $a = 1$ and comparing with our earlier eqs. (3.8) and (3.9) it is thus apparent that $D(b^{-1}y) = D^M(x)$ and that the commutation relations (5.10) and (3.8) are identical. We recall that the relations (3.8) were calculated on the basis of the Lagrangian density (2.9), corresponding to the Fermi gauge for the medium field. It should also be noted that in the case $a = 1$ it follows from (4.10–12) that $L^{\text{vac}}(x) = nL(y)$.

Let us now consider the four-momentum operator for the free radiation field. Since the energy of the field is diagonalizable in the Fermi gauge only [20], we shall for the sake of convenience limit ourselves to this case. However, as has been shown in ref. 20, the corresponding results in the other gauges ($a \neq 1$) can be obtained from the Fermi gauge results by a gauge transformation. The fact that the vacuum operators satisfy the relation

$$i[P_\mu^{\text{vac}}, A_\nu^{\text{vac}}(x)] = \partial_\mu A_\nu^{\text{vac}}(x), \tag{5.13}$$

immediately tells us that the operators

$$P_\mu = (b^{-1})_\mu^\nu P_\nu^{\text{vac}} \tag{5.14}$$

$$\text{satisfy} \quad i[P_\mu, A_\nu(y)] = \partial_\mu^y A_\nu(y) \tag{5.13a}$$

(cf. eq. (2.14a)), and hence should be interpreted as the four-momentum operators for the medium field.

Momentum Space Considerations in the Fermi Gauge

We shall now show that interesting features of the theory are exhibited when we make use of the new covariant polarization vectors in the Fourier expansions. We shall limit ourselves to the Fermi gauge case. The four-potential is expanded in the form

$$A_\mu(x) = \frac{1}{\varrho}\sum_{\boldsymbol{k},\lambda}(b^{-1})_\mu^\nu e_\nu^{(\lambda)}(e^{-ik\cdot x}a^{(\lambda)}(\boldsymbol{k}) + e^{ik\cdot x}\tilde{a}^{(\lambda)}(\boldsymbol{k})), \tag{5.15}$$

where the $e_\nu^{(\lambda)}$ (now assumed to be real) are introduced in eqs. (4.27–36). It must be borne in mind that this simple expansion of the potential corresponds for each $\boldsymbol{k}$ to the effective normalization volume v_k given in eq. (4.22). Note also that in the *rest frame* the spatial vectors $\boldsymbol{e}^{(\lambda)}, \lambda = 1, 2, 3,$

form an orthogonal set normalized to unity, so that the components $a^{(\lambda)}$ give the magnitudes of the potential operator $\boldsymbol{a}$ along each of the orthogonal directions λ. In other inertial frames the vectors $\boldsymbol{e}^{(\lambda)}$ are in general neither orthogonal nor of unit magnitude. The relativistic covariance of the theory is now expressed by requiring a_μ to transform like a four-vector, just as in the classical case. Because of the transformation properties of $e_\mu^{(\lambda)}$ the components $a^{(\lambda)}$ then become *Lorentz invariant* quantities. If we introduce the metric operator η with the properties (3.24) we find, similarly as before, that the total photon number operator for each $\boldsymbol{k}$ is equal to $N_{\text{op}} = -g^{\mu\nu}\tilde{a}_\mu a_\nu = \sum_{\lambda=0}^{3} a^{(\lambda)\dagger} a^{(\lambda)}$. It is clear that the operator N_{op}, as well as the photon number operators for each polarization direction λ separately, $N_{\text{op}}^{(\lambda)} = a^{(\lambda)\dagger} a^{(\lambda)}$, are Lorentz invariants. Further, it turns out that not only the operator components $a^{(\lambda)}$, but also the state vector $|\Psi\rangle$ in the Hilbert (Fock) space, may be assigned a Lorentz invariant meaning. For the Hilbert space in some frame K is spanned by the orthogonal unit vectors $|N^{(\lambda)}(\boldsymbol{k})\rangle$, one vector for each degree of freedom $(\boldsymbol{k},\lambda)$ and a definite value for the occupation number N. In another frame K' we let the Hilbert space be spanned by just the same vectors, but replace each label $N^{(\lambda)}(\boldsymbol{k})$ by a new label $N^{(\lambda)}(\boldsymbol{k}')$, where N and λ are the same and where $\boldsymbol{k}'$ is connected with $\boldsymbol{k}$ by a Lorentz transformation. Since both the partial photon number operators $N_{\text{op}}^{(\lambda)}(\boldsymbol{k})$ and the basic vectors $|N^{(\lambda)}(\boldsymbol{k})\rangle$ in the Hilbert space thus have an invariant meaning it follows that the state vector $|\Psi\rangle$ itself also is a Lorentz invariant.

From the above it is clear that with the use of the new covariant vectors $e_\mu^{(\lambda)}$ we obtain a very convenient description: The Gupta-Bleuler method can be carried through in a completely covariant way in the sense that it involves only *Lorentz invariant operator components and state vectors.* In forming the expectation value of the four-momentum operator of the field we see that the net contribution from the polarization directions $\lambda = 0$ and $\lambda = 1$ vanishes, while the contributions from the directions $\lambda = 2$ and $\lambda = 3$ survive. In any inertial frame the Gupta-Bleuler procedure runs exactly in the same way as in K, although it should be borne in mind that the *direction* of the spatial vector $\boldsymbol{e}^{(\lambda)}$, for a fixed value of λ, depends on the frame K in which the vector is considered. Note that the invariance property of the occupation numbers for each polarization direction λ depends on the use of the *covariant* vectors $e_\mu^{(\lambda)}$; with our old non-covariant vectors defined by (3.20) the invariance property of each $N^{(\lambda)}(\boldsymbol{k})$ is in general lost although also in that case the *total* number $N(\boldsymbol{k})$ remains an invariant.

It should be emphasized that the invariance property of the state vector $|\Psi\rangle$ implies the separate invariance of the state vector $|\Psi_T\rangle$, containing only "transversal" photons for which $\lambda = 2,3$, and the state vectors $|\Phi_{\boldsymbol{k}}\rangle$ containing "longitudinal" and "scalar" photons for which $\lambda = 1,0$, cf. (3.31). In this context we recall that in the cases $\lambda = 1,0$ the occupation numbers are arbitrary except for the single restriction imposed by the gauge condition. Let us elucidate this point by the following simple calculation. A state vector $|\Phi_{\boldsymbol{k}}\rangle$ giving the mixture of longitudinal and scalar photons can be written as [17]

$$|\Phi_{\boldsymbol{k}}\rangle = |\Phi_{\boldsymbol{k}}^{(0)}\rangle + \sum_{N \neq 0} c^{(N)}(\boldsymbol{k})|\Phi^{(N)}(\boldsymbol{k})\rangle, \tag{5.16}$$

where the coefficients $c^{(N)}(\boldsymbol{k}), N = N^{(0)} + N^{(1)}$, are arbitrary quantities. According to the above these coefficients should be *Lorentz invariants*, as we actually shall verify in the case $N = 1$ by calculating the expectation value of the potential $A_\mu(x)$. Assuming that no photon with $\lambda = 2,3$ is present, we get

$$\langle\Psi|\eta A_\mu(x)|\Psi\rangle = \frac{1}{\varrho}\sum_{\boldsymbol{k}}(b^{-1})^\nu_\mu(e_\nu^{(0)} + e_\nu^{(1)})[e^{-ik\cdot x}c^{(1)}(\boldsymbol{k}) + e^{ik\cdot x}c^{(1)*}(\boldsymbol{k})]. \tag{5.17}$$

By inserting the expressions (4.35) and (4.36) we find

$$\langle\Psi|\eta A_\mu(x)|\Psi\rangle = \partial_\mu\chi(x), \tag{5.18a}$$

$$\chi(x) = \frac{i}{n\varrho}\sum_{\boldsymbol{k}}\frac{1}{k\cdot V}[e^{-ik\cdot x}c^{(1)}(\boldsymbol{k}) - e^{ik\cdot x}c^{(1)*}(\boldsymbol{k})]. \tag{5.18b}$$

Eq. (5.18a) tells us that $\chi(x)$ plays the role of a Lorentz invariant gauge function, and eq. (5.18b) requires the expansion coefficients also to be invariants, $c^{(1)}(\boldsymbol{k}) = c^{(1)}(\mathring{\boldsymbol{k}})$, as claimed above. So far only the case $N = 1$ has been considered, but we can verify the invariance of also the other coefficients $c^{(N)}$ by calculating the expectation value of a product of potential components.

It should be said explicitly that the metric operator η itself is an invariant because of the invariance of the individual occupation numbers for ezch λ. In the conventional theory with non-covariant $e_\mu^{(\lambda)}$ this invariance property is lost due to the non-invariance of the scalar photon numbers. The transformation properties of the conventional theory for the vacuum field have been discussed by F. J. Belinfante [21].

We are now in a position to return to a study of the relation (5.14), connecting the four-momenta of the medium field and the vacuum field, in the light of the above developments in Fourier space. To this end let us expand the vacuum field in the form corresponding to (5.15), where $e_\mu^{(\lambda)}$ is the same and where $a^{(\lambda)}(\boldsymbol{k})$ is replaced by $a^{(\lambda)\,\mathrm{vac}}(\boldsymbol{l})$. By means of eq. (4.10) it follows however that $a^{(\lambda)}(\boldsymbol{k}) = a^{(\lambda)\,\mathrm{vac}}(\boldsymbol{l})$ for each pair of $\boldsymbol{k}, \boldsymbol{l}$ that are connected by eq. (4.17). Hence $N_{\mathrm{op}}^{(\lambda)}(\boldsymbol{k}) = N_{\mathrm{op}}^{(\lambda)\,\mathrm{vac}}(\boldsymbol{l})$, and we recover the relation (5.14) by taking into account eq. (4.17). Now it turns out that the equivalence of the medium field and the vacuum field with respect to the operator components applies also with respect to the state vectors. For we may use the same Hilbert (Fock) space in the vacuum case as in the medium case, simply replacing the label $N^{(\lambda)}(\boldsymbol{k})$ of each basic vector $|N^{(\lambda)}(\boldsymbol{k})\rangle$ spanning the Hilbert space by the corresponding label $N^{(\lambda)\,\mathrm{vac}}(\boldsymbol{l})$. Thus we conclude that the state vector $|\Psi\rangle$ of the physical system remains unchanged under the transformation medium field $\rightarrow$ vacuum field, just in the same way as above where we found that $|\Psi\rangle$ remains unchanged under a Lorentz transformation. In particular, the mixture of longitudinal and scalar photons in the medium field is just the same as the mixture in the transformed vacuum field, as expressed by the following property of the expansion coefficients appearing in (5.16): $c^{(N)}(\boldsymbol{k}) = c^{(N)\,\mathrm{vac}}(\boldsymbol{l})$.

We shall finally write down the Feynman rules for the medium field, by starting from the vacuum field case. The propagator in Fourier space for the vacuum field is given by

$$D_{\mu\nu}^{\mathrm{vac}}(l) = -i\frac{g_{\mu\nu}}{l^2 + i\varepsilon}. \tag{5.19}$$

We shall now see that the propagator in configuration space for the medium field can be transformed as

$$D_{\mu\nu}(y - y') = \langle 0|T(A_\mu(y)A_\nu(y'))|0\rangle \tag{5.20 a}$$

$$= \varrho^{-2}(b^{-1})_\mu^\varrho(b^{-1})_\nu^\sigma\langle 0|T(A_\varrho^{\mathrm{vac}}(x)A_\sigma^{\mathrm{vac}}(x'))|0\rangle. \tag{5.20 b}$$

First, it is clear from the remarks above that the ket vector $|0\rangle$ does not change under the transformation from the medium to the vacuum field. Next, we shall see that no difficulty arises from the fact that the time-ordering in (5.20 a) refers to y_0 and y_0' while the time-ordering in (5.20 b) refers to x_0 and x_0'. For by means of the relation

$$T(A_\mu(y)A_\nu(y')) = \tfrac{1}{2}\{A_\mu(y), A_\nu(y')\} + \tfrac{1}{2}\varepsilon(y_0 - y_0')[A_\mu(y), A_\nu(y')] \tag{5.21}$$

it follows that we can verify eq. (5.20 b) by verifying the equation

$$\varepsilon(y_0)D(b^{-1}y) = \varepsilon(x_0)D(x), \tag{5.22}$$

where now the relative distances have been denoted simply by y and x. Because of the D-function in (5.22) the four-vector x_μ must lie on the light cone, i.e. $x^2 = 0$, and then it immediately follows that the vector y_μ is time-like, i.e. $y^2 = (b^2)_{\mu\nu}x^\mu x^\nu > 0$. Thus the sign of x_0, as well as the sign of y_0, are invariants under a Lorentz transformation, and x_0 and y_0 have the same sign in any frame since they have the same sign in the rest frame. This justifies eq. (5.22), and hence also eq. (5.20b).

By means of eqs. (5.19) and (5.20) we now obtain the propagator in momentum space:

$$D_{\mu\nu}(k) = \int dx D_{\mu\nu}(x)e^{ik\cdot x} = -i\mu\frac{(b^{-2})_{\mu\nu}}{k^2+\varkappa(k\cdot V)^2+i\varepsilon}. \tag{5.23}$$

Each internal photon line of four-momentum k_μ in a Feynman diagram may thus be associated with a factor $D_{\mu\nu}(k)$, defined by eq. (5.23).

External photon lines can be handled similarly. We know that an external photon line of four-momentum l_μ and polarization λ in the vacuum case is associated with a factor $e_\mu^{(\lambda)}$; this must be so also in the present case where the $e_\mu^{(\lambda)}$ mean the covariant vectors defined by eqs. (4.27–36). Thus we conclude that the corresponding line in the medium case is associated with a factor $\varrho^{-1}(b^{-1})_\mu^\nu e_\nu^{(\lambda)} = \varrho^{-1}f_\mu^{(\lambda)}$.

The remaining Feynman rules are the same as in the case of a vacuum field. In the practical calculation of transition probabilities it is usually convenient to let the normalization volume $\mathring{\mathscr{V}}$ tend to infinity, in which case the sum over $\boldsymbol{k}$ can be replaced by an integral. An expression for the integration element dm, based upon the expansion (3.6), was given already in eq. (3.16). Now we have in the present section made use of the simple expansion (5.15), which must be associated with the effective normalization volume v_k given by (4.22). This means that we must use the following expression for the integration element dm:

$$dm = \frac{n d\boldsymbol{k}}{(2\pi)^3(1+\varkappa V_0^2)(k_a-k_b)}. \tag{5.24}$$

The above results are consistent with the results obtained by RIAZANOV[(10)]. If we choose to carry through the analysis directly in terms of the medium field, without leaning upon the results from vacuum quantum electrodynamics, we can for instance connect the interacting fields with the non-interacting, incoming fields by means of a unitary transformation (the

U-matrix) in a well known way, and thereby verify the existence of the propagator (5.20a) in the S-matrix.

The Čerenkov effect represents a typical example for an application of the above formalism. We may construct the matrix element for the first order transition and calculate the emission probability in the rest frame of the emitting particle, and will then find agreement with the result obtained by JAUCH and WATSON (6b) except for a factor μ. The extra factor μ in their expression is connected with the fact that their Lagrangian density is defined as μ times the usual Lagrangian density that we have used (cf. eq. (2.9) in the free field case). Consequently, μ will appear also in the Hamiltonian; for instance, their Hamiltonian density in the rest frame $\mathring{K}$ for a free field becomes equal to μ times the usual expression $\frac{1}{2}(\mathring{\boldsymbol{E}}\cdot\mathring{\boldsymbol{D}}+\mathring{\boldsymbol{H}}\cdot\mathring{\boldsymbol{B}})$.

6. Final Remarks

We have seen that the vacuum relations (5.13) are convenient to use in order to find the connection (5.14) between the four-momentum operators. However, the corresponding procedure is not readily performed for the angular momentum operators. This is evidently connected with the fact that the angular momentum for the medium field is not a conserved quantity. Correspondingly, the quantities $M_{\mu\nu}$ do not form a tensor. This last result can be seen most easily by observing that the quantities $M_{\mu\nu}$ defined by (2.19) are physically equivalent to Minkowski's angular momentum based on the tensor expression (2.20), and the latter quantities do not form a tensor since the expressions

$$\partial^{\sigma}(x_{\mu}S^{M}_{\nu\sigma}-x_{\nu}S^{M}_{\mu\sigma}) = S^{M}_{\nu\mu}-S^{M}_{\mu\nu} \tag{6.1}$$

in general do not vanish.

Let us divide the expression (2.18) into two parts and integrate over the volume ($\sigma = 0$):

$$L_{\mu\nu} = \int(x_{\mu}S_{\nu 0}-x_{\nu}S_{\mu 0})\,dV \tag{6.2a}$$

$$\Sigma_{\mu\nu} = \int \pi_{\alpha} I^{\alpha\beta}_{\mu\nu} A_{\beta}\,dV. \tag{6.2b}$$

By inserting the Fourier expansion of the field in these expressions it can be verified that the time-dependence of the spatial components M_{ik} is contained entirely in the part (6.2a), not in the part (6.2b).

For an electromagnetic field in the vacuum [17] it is known that L_{ik} is independent of the polarization possibilities of the photons, except for a term containing the potential component A_0 which is cancelled by a corresponding term in $\sum_{ik}$. We thus get a natural division of the total angular momentum into an orbital part and a spin part. For the medium field the situation is complicated and a corresponding division cannot readily be carried through in the general case. However, the situation of main physical interest arises when the electromagnetic wave runs parallel or antiparallel to the medium velocity (or $\boldsymbol{v} = 0$). In this case L_{ik} and $\sum_{ik}$ are conserved separately; L_{ik} is polarization independent and is interpreted as orbital angular momentum while $\sum_{ik}$ is interpreted as the spin part. In this case we obtain for the constant spin term

$$\textstyle\sum_{23} = \sum\limits_{\boldsymbol{k}} (a_+^\dagger a_+ - a_-^\dagger a_-), \tag{6.3}$$

where as usual

$$a_+ = \frac{1}{\sqrt{2}}(a^{(2)} - ia^{(3)}), \quad a_- = \frac{1}{\sqrt{2}}(a^{(2)} + ia^{(3)}). \tag{6.4}$$

The photons in the medium thus carry spin of unit magnitude.

The next point that we shall comment upon is the connection between the canonical procedure that we have used throughout, and the electromagnetic energy-momentum tensors. We have noted that both the canonical linear momentum and the canonical angular momentum are equivalent to the expressions we obtain by using Minkowski's energy-momentum tensor from the outset. A further illustration of the close connection between Minkowski's tensor and the canonical procedure is provided by eqs. (5.13) and (5.14). This tensor seems on the whole to be the most appropriate one in a quantal treatment, also because of the fact that it corresponds simply to the four-momentum k_μ of a photon in the medium.

It is, however, well known that also other tensor forms have been proposed [4]. It should be emphasized that the above formalism does *not* exclude the legitimacy of other tensor expressions. The best known alternative is due to M. Abraham [2, 4]. This tensor is, however, not divergence-free even for a radiation field, and, as K. Nagy [9] has pointed out, the energy and the momentum cannot be diagonalized simultaneously. From a quantal point of view a more interesting example is the so-called radiation tensor $S_{\mu\nu}^S$ introduced by G. Marx et al. [22], since this tensor is divergence-free and hence implies a conserved four-momentum. We find

$$P_\mu^S = \frac{1}{n^2}\sum_{\boldsymbol{k}}(b^2)_\mu^\nu k_\nu \sum_{\lambda=0}^{3} a^{(\lambda)\dagger} a^{(\lambda)}, \tag{6.5}$$

so that the radiation tensor claims the four-momentum of a photon to be given by $n^{-2}(b^2)_\mu^\nu k_\nu$. As this is a time-like four-vector, the photon energy preserves its sign under proper Lorentz transformations.

When $\boldsymbol{k}\times\boldsymbol{V} = 0$ the spin component corresponding to (6.3) is

$$\sum_{23}^S = \frac{1}{n^2}\sum_{\boldsymbol{k}}(a_+^\dagger a_+ - a_-^\dagger a_-). \tag{6.6}$$

It has however turned out [4] that the radiation tensor is unable to give a simple explanation of the Jones-Richards experiment mentioned in section 1. Now it appears from eqs. (6.3) and (6.6) that an experimental detection of the spin possessed by an electromagnetic wave in a dielectric liquid should yield a further critical test of the phenomenological theory. For instance, if we let a circularly polarized wave be absorbed by a screen immersed in the liquid and then measure the torque exerted on the screen, we may be able to distinguish between the expressions (6.3) and (6.6). Alternatively, instead of letting the incoming wave be absorbed by the screen, we may arrange the equipment so that the screen merely changes the state of polarization of the wave. Such an experiment will thus be a modification of the Carrara experiment [23]. In this context we should also refer to the papers by Toraldo di Francia (ref. 24 with further references therein). As far as we know, this kind of experiment has not been performed. It must be expected that disturbances from the fluid will be an essential difficulty for accurate measurements.

Finally we add some words on the Čerenkov effect. This case exhibits a further characteristic difficulty for the radiation tensor: By calculating the emission angle of a photon in the rest frame $\mathring{K}$ by means of the balance equations for energy and momentum we find, by assigning the spatial momentum $\mathring{\boldsymbol{k}}/n^2$ to the emitted photon in accordance with (6.5), that the emission angle takes a complex non-physical value [4]. Thus the use of the radiation tensor is beset with essential difficulties also from a theoretical point of view.

At the end of the previous section we mentioned that, when applied to the Čerenkov radiation in the rest frame of the emitting particle, the canonical formalism employed in this paper yields a result which is in essential agreement with the Jauch-Watson result. The reason for this agreement does not seem to be straightforward. In the first place, we noted in section 3 some difficulties concerning the Fourier expansions used by JAUCH and WATSON. In the second place, the authors employ a rather complicated Hamiltonian method involving the elimination of the longitudinal field (an extension of the method presented in WENTZEL's book [25]), while our method can be considered to be based upon the application of a unitary transformation connecting the incoming and the outgoing fields. (Whether we choose to transform the results pertaining to the vacuum field or to work directly in terms of the medium field, is immaterial in this context.) These two procedures are apparently rather different and it is not evident that they yield equivalent results in the rest frame of the emitting particle. However, we shall not here pursue this subject further.

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